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A Bochner formula for almost-quaternionic-Hermitian structures

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Abstract

An integral formula is derived, relating the six irreducible components of the intrinsic torsion of an $\mathrm{Sp}_n\mathrm{Sp}_1$ structure on a compact $4n$-dimensional manifold with the Riemann curvature tensor. Some consequences of the formula are studied.

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Introduction

In a previous article [2] we presented a method for obtaining, on a compact manifold with an orthogonal G-structure, an integral formula relating the intrinsic torsion of the structure with the curvature of the underlying Riemannian structure. There, the cases of $G = \mathrm{U}_n$, SU_n, G_2 and Spin_7 were studied. In this follow-up we study the case of $G = \mathrm{Sp}_n\mathrm{Sp}_1$, referred to sometimes in the literature as an "almost-quaternionic-Hermitian structure".

Briefly, the idea of our previous article [2] is the following. Let M be a compact Riemannian manifold with an orthogonal G-structure, i.e., the structure group of M is reduced to a subgroup G of the orthogonal group, where G is assumed to be the stabilizer (in the orthogonal group) of a k-form Φ.

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In the present case of $G = \mathrm{Sp}_n\mathrm{Sp}_1$ we have $k = 4$ and Φ is commonly called "the fundamental 4-form". The covariant derivative $\nabla\Phi$ (with respect to the Levi-Civita connection of the underlying Riemannian metric) can then be naturally identified with the *intrinsic torsion* τ of the G-structure, so that $\nabla\Phi = 0$ if and only if $\tau = 0$, in which case the local holonomy of the Levi-Civita connection is contained in G; see for example the book of S. Salamon [5] for more details.

From the Bochner–Weitzenbock formula for the Laplacian on k-forms one obtains, after integration by parts, a formula of the form

$$\int_M \|d\Phi\|^2 + \|d^*\Phi\|^2 - \|\nabla\Phi\|^2 = \int_M \langle \widetilde{R}\Phi, \Phi\rangle,$$

where $\widetilde{R}$ is a certain operator on k-forms induced by the Riemann curvature tensor R.

Next, using some elementary representation theory, we decompose all tensors in the above formula into their G-irreducible components and obtain, under certain representation-theoretic conditions (satisfied for $G = \mathrm{Sp}_n\mathrm{Sp}_1$; see Section 2.1), a formula relating the L_2-norms of the irreducible components τ_i of the intrinsic torsion with the integral of a certain curvature G-invariant,

$$\sum_i c_i \int_M \|\tau_i\|^2 = \int_M \mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp), \tag{1}$$

for some real constants c_i depending only on G (and neither on M nor on the particular G-structure); $\mathcal{R}$ is the so-called curvature operator of the Riemannian structure (i.e., a section of $\mathrm{End}(\Lambda^2(T^*M))$; see Section 2.1 below for details). In this way, one obtains a curvature obstruction to the existence of certain G-structures characterized by their torsion properties.

This article is devoted to the derivation of the formula in the case of $G = \mathrm{Sp}_n\mathrm{Sp}_1$ and the study of some of its consequences.

In Section 1 following this introduction we collect some standard information about the group $\mathrm{Sp}_n\mathrm{Sp}_1$ and its representations and establish the notation and terminology used in the rest of the article.

Section 2 contains the bulk of the article, consisting of the computation of the constants c_i, thus establishing the precise form of formula (1) (see Theorem 1). This computation recovers the well known fact [8] that, for $n > 2$, an $\mathrm{Sp}_n\mathrm{Sp}_1$-structure with closed fundamental 4-form is torsionless.

Section 3 discusses various consequences of the formula. For example, we derive the following apparently new result (Corollary 2): *A compact quaternionic-Hermitian manifold with non-positive complex sectional curvature is necessarily quaternionic-Kähler.* See Definitions 2 and 3 in Section 3 below for the definitions of complex sectional curvature and quaternionic-Hermitian manifold (respectively).

1. $\mathrm{Sp}_n\mathrm{Sp}_1$-structures

In this section we collect some basic terminology and properties of the group $\mathrm{Sp}_n\mathrm{Sp}_1$ and its representations. We do not claim any originality for this material and suggest the book of S. Salamon [5] and the article of A. Swan [8] as references.

1.1. *Definition of the group* $\mathrm{Sp}_n\mathrm{Sp}_1$

Denote by $\mathbb{H}$ the space of quaternions $x = x_0 + ix_1 + jx_2 + kx_3$, $x_\mu \in \mathbb{R}$, $\mu = 0, \dots, 3$, with $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, etc. Denote conjugation on $\mathbb{H}$ by $x \mapsto \bar{x} = x_0 - ix_1 - jx_2 - kx_3$, so that $|x|^2 = x\bar{x} = \bar{x}x = \sum_{\mu=0}^{3}(x_\mu)^2$ is the usual Euclidean norm. Denote by $\mathbf{V} := \mathbb{H}^n$ the space of columns of n-tuples of quaternions $v = (v_1, \dots, v_n)^t$, $v_\alpha \in \mathbb{H}$, $\alpha = 1, \dots, n$. Introduce a Euclidean norm on $\mathbf{V}$ by $\|v\|^2 := \sum_{\alpha=1}^{n} |v_\alpha|^2$. Then $\mathbf{V}$ is a real $4n$-dimensional Euclidean vector space. Denote its (proper) orthogonal group by SO_{4n}.

Make $\mathbf{V}$ a quaternionic vector space (an $\mathbb{H}$-module) by letting $\mathbb{H}$ act on $\mathbf{V}$ by scalar multiplication on the *right*. The group of $\mathbb{H}$-linear automorphisms of $\mathbf{V}$ is denoted by $\mathrm{GL}_n(\mathbb{H})$, given by left multiplication by $n \times n$ invertible quaternionic matrices.

Right multiplications by i, j, k define on $\mathbf{V}$ three orthogonal almost complex structures I, J, K; denote the corresponding three Kähler forms by $\omega_I, \omega_J, \omega_K$ (respectively).

Let $\mathrm{Sp}_n \subset \mathrm{SO}_{4n}$ denote the subgroup preserving the triple of 2-forms $\omega_I, \omega_J, \omega_K$. An orthogonal transformation preserves an almost complex structure if and only if it preserves the corresponding Kähler form, hence $\mathrm{Sp}_n = \mathrm{SO}_{4n} \cap \mathrm{GL}_n(\mathbb{H})$. In particular, Sp_1 is just unit quaternions.

Let $\mathrm{Sp}_n\mathrm{Sp}_1 \subset \mathrm{SO}_{4n}$ denote the image of $\mathrm{Sp}_n \times \mathrm{Sp}_1$ in SO_{4n} under the combined action on $\mathbf{V}$, $(A, x): v \mapsto Avx^{-1}$. The kernel of this action is easily seen to be $\{\pm(1,1)\} \subset \mathrm{Sp}_n \times \mathrm{Sp}_1$, hence $\mathrm{Sp}_n\mathrm{Sp}_1 \cong \mathrm{Sp}_n \times \mathrm{Sp}_1/\{\pm(1,1)\}$.

Note that $\mathrm{Sp}_1\mathrm{Sp}_1 = \mathrm{SO}_4$, so we will only consider here $\mathrm{Sp}_n\mathrm{Sp}_1$ for $n \geqslant 2$.

1.2. *The fundamental 4-form and the intrinsic torsion*

It is easy to see that the 4-form $\Phi := \omega_I \wedge \omega_I + \omega_J \wedge \omega_J + \omega_K \wedge \omega_K \in \Lambda^4(\mathbf{V}^*)$ is $\mathrm{Sp}_n\mathrm{Sp}_1$-invariant, hence it defines on a $4n$-manifold with an $\mathrm{Sp}_n\mathrm{Sp}_1$-structure a 4-form, called *the fundamental* 4-*form*, and denoted here for simplicity also by Φ.

Moreover, the group $\mathrm{Sp}_n\mathrm{Sp}_1$ is actually the stabilizer of Φ in SO_{4n} (in fact, even in $\mathrm{GL}_{4n}(\mathbb{R})$, for $n \geqslant 2$, although we do not use this fact here), hence a reduction to $\mathrm{Sp}_n\mathrm{Sp}_1$ on a Riemannian $4n$-manifold is given by its fundamental 4-form. The covariant derivative $\nabla\Phi$ can be identified with the intrinsic torsion of the $\mathrm{Sp}_n\mathrm{Sp}_1$-structure, as we now explain.

For a subgroup $G \subset \mathrm{SO}_{4n}$ with a Lie algebra $\mathfrak{g} \subset \mathfrak{so}_{4n} \cong \Lambda^2(\mathbf{V}^*)$, the intrinsic torsion of a G-structure is a section τ of the bundle associated with $W := \mathbf{V}^* \otimes \mathfrak{g}^\perp$, where $\mathfrak{g}^\perp$ is the orthogonal complement of $\mathfrak{g}$ in $\Lambda^2(\mathbf{V}^*)$.

There is a bilinear map $\cdot : \Lambda^2(\mathbf{V}^*) \times \Lambda^k(\mathbf{V}^*) \to \Lambda^k(\mathbf{V}^*)$, essentially the derivative of the pull-back action of SO_{4n} on k-forms, defined by the formula

$$(\theta_1 \wedge \theta_2) \cdot \psi = \theta_2 \wedge \big[\mathrm{int}(\theta_1 \otimes \psi)\big] - \theta_1 \wedge \big[\mathrm{int}(\theta_2 \otimes \psi)\big],$$

where $\mathrm{int}: \mathbf{V}^* \otimes \Lambda^k(\mathbf{V}^*) \to \Lambda^{k-1}(\mathbf{V}^*)$ is "interior product" (contraction), given for $k = 1$ by the inner product, and extended for $k > 1$ as an anti-derivation (with respect to the $\Lambda^k(\mathbf{V}^*)$ factor).

Since $G = \mathrm{Sp}_n\mathrm{Sp}_1$ is the stabilizer of $\Phi \in \Lambda^4(\mathbf{V}^*)$, its Lie algebra $\mathfrak{g} = \mathfrak{sp}_n \oplus \mathfrak{sp}_1$ is the kernel of the map $\cdot\Phi : \Lambda^2(\mathbf{V}^*) \to \Lambda^4(\mathbf{V}^*)$, thus inducing a G-equivariant identification of the torsion space $W := \mathbf{V}^* \otimes \mathfrak{g}^\perp$ with a certain subspace $\widetilde{W} \subset \mathbf{V}^* \otimes \Lambda^4(\mathbf{V}^*)$, mapping $\tau \mapsto \nabla\Phi$. And so in order to decompose $\nabla\Phi$ into its G-irreducible components it suffices to decompose $\mathbf{V}^* \otimes \mathfrak{g}^\perp$ and apply $\cdot\Phi$ to the second factor. This we do in the next subsection.

1.3. Representation theory

In general, the complex irreducible representations of a product of compact groups $G_1 \times G_2$ are given by tensor products $A_1 \otimes A_2$, where A_1 and A_2 are complex irreducible representations of G_1 and G_2 (respectively). When decomposing an $\mathrm{Sp}_n\mathrm{Sp}_1$-representation into irreducibles, we therefore first complexify (in case we start with a real representation such as $\mathbf{V}$), then decompose into a sum of tensor products $A_1 \otimes A_2$, with A_1 and A_2 complex irreducible representations of Sp_n and Sp_1 (respectively). Clearly, as our $G \cong \mathrm{Sp}_n \times \mathrm{Sp}_1/\{\pm(1,1)\}$, we will only encounter $A_1 \otimes A_2$ for which $(-1,-1)$ acts trivially.

Let $\mathbf{E}$ denote the complex vector space obtained from $\mathbf{V} = \mathbb{H}^n$ by fixing the almost-complex structure I (i.e., restrict the right $\mathbb{H}$-action to $\mathbb{C} \subset \mathbb{H}$). Then left multiplication by quaternionic matrices turns $\mathbf{E}$ into a complex $2n$-dimensional irreducible unitary representation space for Sp_n.

Let $e_1, \ldots, e_n$ be a quaternionic unitary basis for $\mathbf{V}$ (i.e., they are mutually orthogonal unitary vectors which $\mathbb{H}$-span $\mathbf{V}$) and let $e^\alpha = e_\alpha j$, $\alpha = 1, \ldots, n$. Then $\{e_\alpha, e^\alpha\}_{\alpha=1}^n$ is a (complex) unitary basis for $\mathbf{E}$. Denote by $\{z_\alpha, z^\alpha\}_{\alpha=1}^n$ the corresponding (complex) dual basis of $\mathbf{E}^*$. Define $\Omega = \omega_J - i\omega_K$. A computation shows that $\Omega = \sum_\alpha z_\alpha \wedge z^\alpha$. Hence $\Omega \in \Lambda^2(\mathbf{E}^*)$ and is Sp_n-invariant.

Denote the orthogonal complement of Ω in $\Lambda^2(\mathbf{E}^*)$ by $\Lambda_0^2(\mathbf{E}^*)$. More generally, denote the orthogonal complement of $\Omega \wedge \Lambda^{k-2}(\mathbf{E}^*)$ in $\Lambda^k(\mathbf{E}^*)$ by $\Lambda_0^k(\mathbf{E}^*)$. Then $\mathbf{E}^*$, $\Lambda_0^2(\mathbf{E}^*)$, $\Lambda_0^3(\mathbf{E}^*), \ldots, \Lambda_0^n(\mathbf{E}^*)$ are complex irreducible, mutually distinct, Sp_n-representations.

Passing to Sp_1, we denote by $\boldsymbol{\Sigma}$ the dual of the complex 2-dimensional Sp_1-representation obtained from $\mathbb{H}$ by restricting to right-scalar multiplication by $\mathbb{C} \subset \mathbb{H}$. Let $\{p, q\} \subset \boldsymbol{\Sigma}$ be the basis dual to $\{1, j\}$. Then $\omega := p \wedge q \in \Lambda^2(\boldsymbol{\Sigma})$ is Sp_1-invariant. A complete list of the complex irreducible representations of Sp_1 is given by the symmetric powers $\boldsymbol{\Sigma}^k := S^k(\boldsymbol{\Sigma})$, $k = 0, 1, 2, \ldots$.

Next, we have an isomorphism of complex $\mathrm{Sp}_n\mathrm{Sp}_1$ representations, $\mathbf{E} \otimes_{\mathbb{C}} \boldsymbol{\Sigma}^* \cong \mathbf{V} \otimes_{\mathbb{R}} \mathbb{C}$, given on basis elements by

$$
\begin{aligned}
e_\alpha \otimes 1 &\mapsto e_\alpha - \sqrt{-1}(e_\alpha i), & e_\alpha \otimes j &\mapsto e_\alpha j - \sqrt{-1}(e_\alpha k), \\
e^\alpha \otimes 1 &\mapsto e_\alpha j + \sqrt{-1}(e_\alpha k), & e^\alpha \otimes j &\mapsto -e_\alpha - \sqrt{-1}(e_\alpha i),
\end{aligned}
$$

followed by multiplication by $1/\sqrt{2}$ (so as to be an isometry).

Using this isomorphism, we have

$$
\begin{aligned}
\Lambda^2(\mathbf{V}^*) \otimes \mathbb{C} \cong \Lambda^2(\mathbf{E}^* \otimes \boldsymbol{\Sigma}) &= \left[S^2(\mathbf{E}^*) \otimes \Lambda^2(\boldsymbol{\Sigma})\right] \oplus \left[\Lambda^2(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}^2\right] \\
&= \left[S^2(\mathbf{E}^*) \otimes \omega\right] \oplus [\Omega \otimes \boldsymbol{\Sigma}^2] \oplus \left[\Lambda_0^2(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}^2\right].
\end{aligned}
$$

The first two summands in the last formula correspond to the Lie algebra $\mathfrak{g} := \mathfrak{sp}_n \oplus \mathfrak{sp}_1 \subset \mathfrak{so}_{4n} \cong \Lambda^2(\mathbf{V}^*)$ so the last summand is $\mathfrak{g}^\perp \otimes \mathbb{C}$ and is irreducible.

We thus get for the $\mathrm{Sp}_n\mathrm{Sp}_1$ intrinsic torsion space

$$
\begin{aligned}
W \otimes \mathbb{C} := (\mathbf{V}^* \otimes \mathfrak{g}^\perp) \otimes \mathbb{C} &\cong [\mathbf{E}^* \otimes \boldsymbol{\Sigma}] \otimes \left[\Lambda_0^2(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}^2\right] \\
&\cong \left[\mathbf{E}^* \otimes \Lambda_0^2(\mathbf{E}^*)\right] \otimes [\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2].
\end{aligned} \tag{2}
$$

Now we need the following decompositions:

- The Sp_n-decomposition:

$$
\mathbf{E}^* \otimes \Lambda_0^2(\mathbf{E}^*) \cong \Lambda_0^3(\mathbf{E}^*) \oplus \mathbf{E}^* \oplus \mathbf{K},
$$

where
- $\Lambda^3_0(\mathbf{E}^*) \to \mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)$ is given by inclusion; for $n = 2$, $\Lambda^3_0(\mathbf{E}^*) = 0$.
- $\mathbf{E}^* \to \mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)$ is given by wedging with Ω followed by orthogonal projection $\mathbf{E}^* \otimes \Lambda^2(\mathbf{E}^*) \to \mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)$;
- $\mathbf{K} \subset \mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)$ is the kernel of the "Bianchi" symmetrizer $1 + (123) + (132)$, i.e., the space of all tensors $T \in \mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)$ satisfying the identity $T(e_1, e_2, e_3) + T(e_2, e_3, e_1) + T(e_3, e_1, e_2) = 0$, for all $e_1, e_2, e_3 \in \mathbf{E}$. Another description of $\mathbf{K}$, in terms of Young symmetrizers, is as the image of $\mathbf{E}^* \otimes \mathbf{E}^* \otimes \mathbf{E}^*$ under $(1 - (23))(1 + (12))$, followed by the projection $\mathbf{E}^* \otimes \Lambda^2(\mathbf{E}^*) \to \mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)$.

- The Sp_1-decomposition:

$$\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2 \cong \boldsymbol{\Sigma} \oplus \boldsymbol{\Sigma}^3,$$

where
- $\boldsymbol{\Sigma} \to \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2$ is given by tensoring with ω, $\theta \mapsto \omega \otimes \theta$, followed by orthogonal projection on $\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2$ (symmetrization in the second and third entries).
- $\boldsymbol{\Sigma}^3 \to \boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2$ is given by inclusion.

The above information, once inserted into formula (2), yields

Proposition 1. *The* $\mathrm{Sp}_n\mathrm{Sp}_1$ *torsion space* $W := \mathbf{V}^* \otimes \mathfrak{g}^\perp$ *decomposes into the direct sum of* 6 *irreducible non-isomorphic subspaces, corresponding to the* 6 *summands one gets after expanding the right-hand side of*

$$[\mathbf{V}^* \otimes \mathfrak{g}^\perp] \otimes \mathbb{C} \cong \left[\mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)\right] \otimes [\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2] \cong \left[\Lambda^3_0(\mathbf{E}^*) \oplus \mathbf{E}^* \oplus \mathbf{K}\right] \otimes [\boldsymbol{\Sigma} \oplus \boldsymbol{\Sigma}^3].$$

Let us denote these 6 irreducible summands of the torsion space by $W_1, \ldots, W_6$,

$$W_1 \otimes \mathbb{C} \cong \Lambda^3_0(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}^3, \qquad W_2 \otimes \mathbb{C} \cong \mathbf{E}^* \otimes \boldsymbol{\Sigma}^3, \qquad W_3 \otimes \mathbb{C} \cong \mathbf{K} \otimes \boldsymbol{\Sigma}^3,$$
$$W_4 \otimes \mathbb{C} \cong \Lambda^3_0(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}, \qquad W_5 \otimes \mathbb{C} \cong \mathbf{E}^* \otimes \boldsymbol{\Sigma}, \qquad W_6 \otimes \mathbb{C} \cong \mathbf{K} \otimes \boldsymbol{\Sigma}.$$

Note that since the 6 summands are non-isomorphic, they must be mutually orthogonal.

Finally, note that for $n = 2$, since $\Lambda^3_0(\mathbf{E}^*) = 0$, there are only 4 irreducible summands (omitting W_1 and W_4).

2. The $\mathrm{Sp}_n\mathrm{Sp}_1$ Bochner formula

2.1. The Bochner formula for orthogonal G-structures

Let us recall from our previous article [2] the general Bochner type formula for an orthogonal G-structure on a compact manifold, where G is the stabilizer of a k-form Φ:

$$\int_M \|d\Phi\|^2 + \|\delta\Phi\|^2 - \|\nabla\Phi\|^2 = \int_M \langle \widetilde{R}\Phi, \Phi\rangle, \tag{3}$$

where $\widetilde{R}$ is the operator on k-forms obtained from the Riemann curvature tensor R as follows: consider R as a section of $\Lambda^2(M) \otimes \Lambda^2(M)$, $R = \sum \alpha \otimes \beta$, then $\widetilde{R}\Phi = \sum \alpha \cdot (\beta \cdot \Phi)$.

Next, we make the following assumptions on G:

(i) $\mathfrak{g}^\perp$ is irreducible.
(ii) $W = \mathbf{V}^* \otimes \mathfrak{g}^\perp$ is multiplicity free.

Note: both assumption are satisfied for our group $G = \mathrm{Sp}_n\mathrm{Sp}_1$ (see Section 1.3).

With these assumptions, one decomposes $W = \bigoplus_{i=1}^r W_i$, where the W_i are G-irreducible and pairwise non-isomorphic (by assumption (ii)) and accordingly $\nabla\Phi = \sum_{i=1}^r (\nabla\Phi)_i$, with $(\nabla\Phi)_i \in \widetilde{W}_i$, where $\widetilde{W}_i$ is the image of W_i under the embedding $W = \mathbf{V}^* \otimes \mathfrak{g}^\perp \to \mathbf{V}^* \otimes \Lambda^k(V^*)$, $\theta \otimes \beta \mapsto \theta \otimes (\beta \cdot \Phi)$. Since the $\widetilde{W}_i$ are irreducible and non-isomorphic they are mutually orthogonal, hence $\|\nabla\Phi\|^2 = \sum_{i=1}^r \|(\nabla\Phi)_i\|^2$.

From $\nabla\Phi$ one obtains $d\Phi = \mathrm{alt}(\nabla\Phi)$ and $\delta\Phi = -\mathrm{int}(\nabla\Phi)$ by the linear maps $\mathrm{alt}: \mathbf{V}^* \otimes \Lambda^k(\mathbf{V}^*) \to \Lambda^{k+1}(\mathbf{V}^*)$ (exterior product, or *alternation*) and $\mathrm{int}: \mathbf{V}^* \otimes \Lambda^k(\mathbf{V}^*) \to \Lambda^{k-1}(\mathbf{V}^*)$ as in Section 1.2.

When restricting the G-equivariant maps alt and int to the irreducible summands $\widetilde{W}_i$ they must be a homothety onto their image (by Schur's lemma), hence there exist non-negative constants a_i, b_i, such that $\|\mathrm{alt}(\tilde{w}_i)\|^2 = a_i\|\tilde{w}_i\|^2$, $\|\mathrm{int}(\tilde{w}_i)\|^2 = b_i\|\tilde{w}_i\|^2$, for all $\tilde{w}_i \in \widetilde{W}_i$, $i = 1, \ldots, r$. It follows that $\|d\Phi\|^2 = \sum_{i=1}^r a_i\|(\nabla\Phi)_i\|^2$ and $\|d^*\Phi\|^2 = \sum_{i=1}^r b_i\|(\nabla\Phi)_i\|^2$.

Let $\tau = \sum \tau_i$ be the decomposition of the intrinsic torsion into irreducibles, $\tau_i \in W_i$. Then, by assumption (i), the map $V^* \otimes \mathfrak{g}^\perp \to \widetilde{W}$, $\tau \mapsto \nabla\Phi$, is a homothety, hence there is a constant $C > 0$ such that $\|(\nabla\Phi)_i\|^2 = C\|\tau_i\|^2$.

Regarding the curvature term on the right hand side of formula (3), we recall from [2] the following calculation:

$$\langle \widetilde{R}\Phi, \Phi\rangle = \sum \bigl\langle \alpha \cdot (\beta \cdot \Phi), \Phi\bigr\rangle = -\sum \langle \beta \cdot \Phi, \alpha \cdot \Phi\rangle = C\,\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp\rangle,$$

where $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp\rangle$ denotes "the trace of the $(\mathfrak{g}^\perp, \mathfrak{g}^\perp)$ block" of the curvature operator (the latter is R interpreted as an endomorphism of $\Lambda^2(\mathbf{V}^*)$; note also an annoying switching of signs between R and $\mathcal{R}$ which we are unable to avoid).

In this way, after we determine the homothety factors a_i, b_i (in the next subsection), formula (3) becomes

$$\sum_{i=1}^r c_i \int_M \|\tau_i\|^2 = \int_M \mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp), \tag{4}$$

with $c_i = a_i + b_i - 1$.

2.2. *The homothety factors a_i, b_i for $G = \mathrm{Sp}_n\mathrm{Sp}_1$*

For each $i = 1, \ldots, 6$ we pick a non-zero element $w_i \in W_i \otimes \mathbb{C}$, determine its image $\tilde{w}_i \in \widetilde{W}_i \otimes \mathbb{C}$, apply alt and int, and calculate norms. The outcome of this calculation is given in the following tables. In the next subsection we give some information on the calculations involved in obtaining Tables 1 and 2.

Remarks.

1. For $n \geqslant 3$, it follows immediately from the fact that all the $a_i \neq 0$ that the fundamental 4-form Φ is parallel if it is closed. This has already been noticed before by Swan [8].

2. The case $n = 2$ is different from $n \geqslant 3$ in two respects: first, the components $(\nabla\Phi)_1$ and $(\nabla\Phi)_4$ are absent; and second, of the 4 remaining terms, the component $\mathrm{alt}((\nabla\Phi)_3)$ vanishes identically. Consequently, the vanishing of $d\Phi$ is not sufficient in general to guarantee the vanishing of $\nabla\Phi$. In fact,

Table 1
Summary of calculations for $n \geqslant 3$

	w_i	$\|\tilde{w}_i\|^2$	$\|\operatorname{alt}(\tilde{w}_i)\|^2$	$\|\operatorname{int}(\tilde{w}_i)\|^2$	a_i	b_i
W_1	$a \otimes s$	$12n$	$15(2n-1)$	9	$\frac{5(2n-1)}{4n}$	$\frac{3}{4n}$
W_2	$b \otimes s$	$4(n-1)(2n+1)$	$5(n-1)(2n+1)$	$(n-1)(2n+1)^2$	$\frac{5}{4}$	$\frac{2n+1}{4}$
W_3	$c \otimes s$	$4n$	$n-2$	0	$\frac{n-2}{4n}$	0
W_4	$a \otimes t$	$18n$	$18(n+1)$	0	$\frac{n+1}{n}$	0
W_5	$b \otimes t$	$6(n-1)(2n+1)$	$12(n-1)(2n+1)$	$12(n-1)^2(2n+1)$	2	$2(n-1)$
W_6	$c \otimes t$	$6n$	$6n-3$	9	$\frac{2n-1}{2n}$	$\frac{3}{2n}$

Table 2
Summary of calculations for $n = 2$

	w_i	$\|\tilde{w}_i\|^2$	$\|\operatorname{alt}(\tilde{w}_i)\|^2 = \|\operatorname{int}(\tilde{w}_i)\|^2$	$a_i = b_i$
W_2	$b \otimes s$	20	25	5/4
W_3	$c \otimes s$	8	0	0
W_5	$b \otimes t$	30	60	2
W_6	$c \otimes t$	12	9	3/4

Salamon [6] has recently constructed a compact 8-manifold carrying a non-parallel $\mathrm{Sp}_2\mathrm{Sp}_1$-structure with closed Φ.

As a consequence of the calculation we get the following:

Theorem 1. *Let M be a compact $4n$-dimensional manifold, $n \geqslant 2$, with an $\mathrm{Sp}_n\mathrm{Sp}_1$-structure with an intrinsic torsion τ. Let $\tau = \sum_{i=1}^{6} \tau_i$ be the decomposition of τ into irreducible components (see Proposition 1; note that for $n = 2$, $\tau_1 = \tau_4 = 0$). Set $E_i = \int_M \|\tau_i\|^2$, $i = 1, \dots, 6$. Let $\operatorname{tr}(\mathcal{R}, \mathfrak{g}^\perp)$ be the trace of the Riemann curvature operator of M restricted to the orthogonal complement $\mathfrak{g}^\perp$ of the Lie algebra of $\mathrm{Sp}_n\mathrm{Sp}_1$ in $\Lambda^2(\mathbf{V}^*)$, followed by orthogonal projection onto $\mathfrak{g}^\perp$. Then for all $n \geqslant 3$*

$$\frac{3n-1}{2n}E_1 + \frac{n+1}{2}E_2 - \frac{3n+2}{4n}E_3 + \frac{1}{n}E_4 + (2n-1)E_5 + \frac{1}{n}E_6 = \int_M \operatorname{tr}(\mathcal{R}, \mathfrak{g}^\perp).$$

For $n = 2$ the formula is

$$\frac{3}{2}E_2 - E_3 + 3E_5 + \frac{1}{2}E_6 = \int_M \operatorname{tr}(\mathcal{R}, \mathfrak{g}^\perp).$$

2.3. Comments regarding the calculation of a_i, b_i

1. Denote the basis elements $z_\alpha \otimes p, z^\alpha \otimes p, z_\alpha \otimes q, z^\alpha \otimes q$ of $\mathbf{E}^* \otimes \boldsymbol{\Sigma} \cong \mathbf{V}^* \otimes \mathbb{C}$ by $p_\alpha, p^\alpha, q_\alpha, q^\alpha$ (respectively). In terms of this basis, the ($\mathbb{C}$-bilinear) inner-product is given by $\langle p_\alpha, q^\alpha \rangle = 1$, $\langle p^\alpha, q_\alpha \rangle = -1$, and the remaining pairs of elements of the basis are orthogonal.

2. Denote basis elements of $\Lambda^*(\mathbf{E}^*)$ by $z_{\alpha\beta} := z_\alpha \wedge z_\beta$, $z_\alpha^\beta := z_\alpha \wedge z^\beta$, $z_{\alpha\beta}^\gamma := z_\alpha \wedge z_\beta \wedge z^\gamma$ (in this order), $\ldots$etc. Similarly, denote basis elements of $\Lambda^*(\mathbf{E}^* \otimes \boldsymbol{\Sigma}) \cong \Lambda^*(\mathbf{V}^*) \otimes \mathbb{C}$ by $p_{\alpha\beta} = p_\alpha \wedge p_\beta$, $p_\alpha^\beta = p_\alpha \wedge p^\beta$, $p_{\alpha\beta}^\gamma := p_\alpha \wedge p_\beta \wedge p^\gamma$, $\ldots$etc.

3. Omit the $\otimes$ symbol; e.g., $p^2 = p \otimes p \in \boldsymbol{\Sigma}^2$, $z_1 z_2^3 = z_1 \otimes (z_2 \wedge z^3)$, $\ldots$etc.

4. Define the following elements in the three irreducible summands of $\mathbf{E}^* \otimes \Lambda_0^2(\mathbf{E}^*)$:

- In the $\Lambda_0^3(\mathbf{E}^*)$ summand, for $n \geqslant 3$, let

$$a := z_1 z_{23} + z_2 z_{31} + z_3 z_{12};$$

i.e., $a = z_{123}$ up to a constant.

- In the $\mathbf{E}^*$ summand, let

$$b := z_1 \Omega / n + \sum_\alpha z^\alpha z_{1\alpha} - z_\alpha z_1^\alpha.$$

This we get by starting with $z_1 \wedge \Omega = \text{const.}(z_1 \Omega + \sum_a [z^\alpha z_{1\alpha} - z_\alpha z_1^\alpha]) \in \mathbf{E}^* \otimes \Lambda^2(\mathbf{E}^*)$, then apply, in the $\Lambda^2(\mathbf{E}^*)$ factor, orthogonal projection onto $\Lambda_0^2(\mathbf{E}^*)$. Using the Hermitian inner product $h(\cdot\,,\cdot)$, this projection is $\beta \mapsto \beta - \frac{h(\beta,\Omega)}{h(\Omega,\Omega)}\Omega$. Now $h(\Omega, \Omega) = \sum_{\alpha,\beta} h(z_\alpha^\alpha, z_\beta^\beta) = n$, and so $z_1 \Omega \mapsto 0$, $\sum_a z^\alpha z_{1\alpha} \mapsto \sum_a z^\alpha z_{1\alpha}$ and $-\sum_a z_\alpha z_1^\alpha \mapsto \sum_a z_\alpha h(z_1^\alpha, \Omega)\Omega/n - z_\alpha z_1^\alpha = z_1 \Omega / n - \sum_a z_\alpha z_1^\alpha$, from which the value of b follows.

- In the $\mathbf{K}$ summand, let

$$c := z_1 z_{12}.$$

This is obtained by applying the Young symmetrizer $(1 - (23))(1 + (12))$ to $z_1 z_1 z_2$, followed by orthogonal projection onto $\mathbf{E}^* \otimes \Lambda_0^2(\mathbf{E}^*)$, as described before Proposition 1.

5. Define the following elements in the irreducible summands of the decomposition of $\boldsymbol{\Sigma} \otimes \boldsymbol{\Sigma}^2$:

- In the $\boldsymbol{\Sigma}^3$ summand: let

$$s = p^3.$$

- In the $\boldsymbol{\Sigma}$ summand: let

$$t := p(pq + qp)/2 - qp^2.$$

This we get by applying the process described before Proposition 1 to $\omega p = (pq - qp)p$.

6. For each of the torsion space elements $w_1 = a \otimes s$, $w_2 = b \otimes s$, $\ldots$, as defined above, we need to find a corresponding element $\tilde{w}_i \in \widetilde{W}_i \otimes \mathbb{C} \subset \mathbf{V}^* \otimes (\mathfrak{g}^\perp \cdot \Phi) \otimes \mathbb{C}$. For this, one needs in principle to write explicitly Φ and apply $\cdot\Phi : \mathfrak{g}^\perp \to \Lambda^4(\mathbf{V}^*)$ to the second factor in $\mathbf{V}^* \otimes \mathfrak{g}^\perp$. However, we found that it was easier to "guess" the outcome of this map. The point is that *any* non-zero G-equivariant map $\mathfrak{g}^\perp \to \Lambda^4(\mathbf{V}^*)$ will do: one can verify first that the irreducible G-representation $\mathfrak{g}^\perp \otimes \mathbb{C} \cong \Lambda_0^2(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}^2$ appears with multiplicity 1 in $\Lambda^4(\mathbf{V}^*) \otimes \mathbb{C} \cong \Lambda^4(\mathbf{E}^* \otimes \boldsymbol{\Sigma})$; hence, by Schur's lemma, any two G-equivariant maps $\mathfrak{g}^\perp \otimes \mathbb{C} \to \Lambda^4(\mathbf{V}^*) \otimes \mathbb{C}$ coincide, up to a constant. We proceed to give such a map as a composition of

"obvious" maps as follows:

$$\begin{aligned}\Lambda^2_0(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2 &\xrightarrow{f_1}\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2\xrightarrow{f_2}\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2\\ &\xrightarrow{f_3}\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\boldsymbol{\Sigma}^2)\xrightarrow{f_4}\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2\otimes\boldsymbol{\Sigma}^2\\ &\xrightarrow{f_5}\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2\otimes\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2\xrightarrow{f_6}\Lambda^2(\mathbf{E}^*\otimes\boldsymbol{\Sigma})\otimes\Lambda^2(\mathbf{E}^*\otimes\boldsymbol{\Sigma})\\ &\xrightarrow{f_7}\Lambda^4(\mathbf{E}^*\otimes\boldsymbol{\Sigma}),\end{aligned}$$

where

- f_1 is given by the inclusion $\Lambda^2_0(\mathbf{E}^*)\to\Lambda^2(\mathbf{E}^*)$ tensored with the identity map on $\boldsymbol{\Sigma}^2$;
- f_2 is given by inserting the Sp_n-invariant $\Omega=\sum z^\alpha_\alpha$ in the second factor of $\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2$;
- f_3 is given by the identity map on $\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\mathbf{E}^*)$ tensored with an Sp_1-isomorphism $\boldsymbol{\Sigma}^2\to\Lambda^2(\boldsymbol{\Sigma}^2)$ (essentially the Hodge isomorphism; note that $\boldsymbol{\Sigma}^2$ is 3-dimensional):

$$\begin{aligned}&p^2\mapsto p^2(pq+qp)-(pq+qp)p^2,\\ &pq+qp\mapsto 2(p^2q^2-q^2p^2),\\ &q^2\mapsto(pq+qp)q^2-q^2(pq+qp);\end{aligned}$$

- f_4 is given by the identity map on $\Lambda^2(\mathbf{E}^*)\otimes\Lambda^2(\mathbf{E}^*)$ tensored with the inclusion $\Lambda^2(\boldsymbol{\Sigma}^2)\to\boldsymbol{\Sigma}^2\otimes\boldsymbol{\Sigma}^2$;
- f_5 is given by interchanging the second and the third factor;
- f_6 is given by the inclusion $\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2\to\Lambda^2(\mathbf{E}^*\otimes\boldsymbol{\Sigma})=[\Lambda^2(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2]\oplus[S^2(\mathbf{E}^*)\otimes\Lambda^2(\boldsymbol{\Sigma})]$ tensored with itself;
- f_7 is given by antisymmetrization.

Each of these maps is clearly $\mathrm{Sp}_n\mathrm{Sp}_1$-equivariant, hence their composition is also, therefore it is a constant multiple of the (complexification of the) desired map $\cdot\Phi:\mathfrak{g}^\perp\to\Lambda^4(\mathbf{V}^*)$.

Thus, for example, if we start with $p_{12}=z_{12}p^2\in\Lambda^2_0(\mathbf{E}^*)\otimes\boldsymbol{\Sigma}^2$ we obtain

$$\begin{aligned}z_{12}p^2&\overset{f_2\circ f_1}{\longmapsto}\sum_\alpha z_{12}z^\alpha_\alpha p^2\\ &\overset{f_4\circ f_3}{\longmapsto}\sum_\alpha z_{12}z^\alpha_\alpha\big[p^2(pq+qp)-(pq+qp)p^2\big]\\ &\overset{f_5}{\longmapsto}\sum_\alpha z_{12}\big[p^2z^\alpha_\alpha(pq+qp)-(pq+qp)z^\alpha_\alpha p^2\big]\\ &\overset{f_6}{\longmapsto}\sum_\alpha\big[p_{12}(p_\alpha q^\alpha+q_\alpha p^\alpha)-(p_1q_2+q_1p_2)p^\alpha_\alpha\big]\\ &\overset{f_7}{\longmapsto}\sum_\alpha(p_{12\alpha}\wedge q^\alpha-p^\alpha_{12}\wedge q_\alpha-p^\alpha_{1\alpha}\wedge q_2+p^\alpha_{2\alpha}\wedge q_1).\end{aligned}$$

As another example, take $p_1\wedge q_2+q_1\wedge p_2=z_{12}(pq+qp)$, obtaining

$$z_{12}(pq+qp)\overset{f_4\circ\cdots\circ f_1}{\longmapsto}\sum_\alpha 2z_{12}z^\alpha_\alpha(p^2q^2-q^2p^2)\mapsto 2\sum_\alpha(p_{12}\wedge q^\alpha_\alpha-q_{12}\wedge p^\alpha_\alpha).$$

7. To calculate the norms of the $\tilde{w}_i$ it is actually simpler to calculate the norm of the $w_i \in \mathbf{V}^* \otimes \mathfrak{g}^\perp$ and multiply by the homothety factor C of our map $\mathfrak{g}^\perp \to \Lambda^4$. From either of the above examples one can calculate this factor: taking $z_{12}(pq+qp)$, we have

$$\left\| z_{12}(pq+qp) \right\|^2 = \|z_{12}\|^2 \|pq+qp\|^2 = 1 \cdot 2 = 2,$$
$$\left\| 2 \sum_\alpha (p_{12} \wedge q^\alpha_\alpha - q_{12} \wedge p^\alpha_\alpha) \right\|^2 = 4(n+n) = 8n,$$

hence we get that the factor is $C = 8n/2 = 4n$.

8. The zeros in the table are explained by showing that $\Lambda^3(\mathbf{V}^*)$ does not contain irreducible summands of type W_3 or W_4.

9. Now we need to calculate for each of the 6 elements $w_i \in W_i \otimes \mathbb{C}$, the corresponding element $\tilde{w}_i \in \widetilde{W}_i \otimes \mathbb{C} \subset \mathbf{V}^* \otimes \Lambda^4(\mathbf{V}^*)$, then the norms of $\tilde{w}_i$, $\mathrm{alt}(\tilde{w}_i)$ and $\mathrm{int}(\tilde{w}_i)$. This is not a particularly pleasant task, even after all the above remarks and shortcuts. We shall present the calculation only for the first element $w_1 = a \otimes s$, after which the reader would rather check the other cases more efficiently by herself than follow our detailed presentation.

So if we start with $w_1 = a \otimes s$ we end up with the following element $\tilde{w}_1$:

$$\begin{aligned} a \otimes s &= (z_1 z_{23}) p^3 + \cdots \text{etc.} \mapsto p_1(z_{23}p^2) + \cdots \text{etc.} \\ &\mapsto \sum_a p_1(p_{23\alpha} \wedge q^\alpha - p^\alpha_{23} \wedge q_\alpha - p^\alpha_{2\alpha} \wedge q_3 + p^\alpha_{3\alpha} \wedge q_2) + \cdots \text{etc.} \\ &= \tilde{w}_1, \end{aligned}$$

where "$\cdots$ etc." stands for 2 more similar terms obtained by cyclic permutations of 1, 2, 3.

We thus get

$$\begin{aligned} \mathrm{alt}(\tilde{w}_1) &= \sum_\alpha (p_{123\alpha} \wedge q^\alpha - p^\alpha_{123} \wedge q_\alpha - p^\alpha_{12\alpha} \wedge q_3 + p^\alpha_{13\alpha} \wedge q_2) + \cdots \text{etc.} \\ &= -5 p_{123} \wedge (p^1 \wedge q_1 + \cdots \text{etc.}) \\ &\quad + \sum_{\alpha \geqslant 4} \big[3 p_{123} \wedge (p_\alpha \wedge q^\alpha - p^\alpha \wedge q_\alpha) - 2 p^\alpha_\alpha \wedge (p_{12} \wedge q_3 + \cdots \text{etc.}) \big], \\ \mathrm{int}(\tilde{w}_1) &= 3 p_{123}. \end{aligned}$$

Hence

$$\|\tilde{w}_1\|^2 = 4n \|w_1\|^2 = 4n \cdot 3 = 12n,$$
$$\left\| \mathrm{alt}(\tilde{w}_1) \right\|^2 = 25 \cdot 3 + 9 \cdot 2(n-3) + 4(n-3)3 = 15(2n-1),$$
$$\left\| \mathrm{int}(\tilde{w}_1) \right\|^2 = 9,$$

and

$$a_1 = \frac{15(2n-1)}{12n} = \frac{5(2n-1)}{4n}, \qquad b_1 = \frac{9}{12n} = \frac{3}{4n}.$$

10. For $n = 2$, we have the identity $\|\mathrm{alt}(\tilde{w})\|^2 = \|\mathrm{int}(\tilde{w})\|^2$, $\tilde{w} \in \widetilde{W}$. This follows from the (anti-)self-duality of the 4-form Φ: use the identity $*(\theta \wedge \psi) = \mathrm{int}(\theta \otimes *\psi)$, holding for any 1-form θ and p-form

ψ; applied to $\tilde{w} = \theta \otimes (\beta \cdot \Phi)$, $\beta \in \mathfrak{g}^\perp$, get $*[\mathrm{alt}(\tilde{w})] = \mathrm{int}[\theta \otimes *(\beta \cdot \Phi)] = \mathrm{int}[\theta \otimes (\beta \cdot *\Phi)] = \pm\,\mathrm{int}(\tilde{w})$, hence $\|\mathrm{alt}(\tilde{w})\| = \|\mathrm{int}(\tilde{w})\|$. A quick representation theoretic proof of the (anti-)self-duality of Φ, without an explicit calculation of Φ, consists of verifying that the trivial subspace (G-fixed) of $\Lambda^4(\mathbf{V}^*)$ is 1-dimensional. Since the Hodge star commutes with the G-action we have that $*\Phi = c\Phi$; but $*$ is an isometry, hence c must be ± 1.

3. Applications

Definition 1. An $\mathrm{Sp}_n\mathrm{Sp}_1$-structure with vanishing torsion, $\tau = 0$, is called quaternionic-Kähler.

All the applications we shall present here are based on the following obvious consequence of Theorem 1:

Corollary 1. *Let M be a $4n$-dimensional compact manifold with an $\mathrm{Sp}_n\mathrm{Sp}_1$-structure such that $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp) \leqslant 0$ and $\tau_3 = 0$, or $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp) \geqslant 0$ and $\tau = \tau_3$ (i.e., $\tau_1 = \tau_2 = \tau_4 = \tau_5 = \tau_6 = 0$ for $n \geqslant 3$, or $\tau_2 = \tau_5 = \tau_6 = 0$ for $n = 2$). Then the structure is in fact quaternionic-Kähler.*

We shall now study the conditions appearing in the above corollary.

Definition 2. A Riemannian manifold $(M, \langle\,,\rangle)$ is said to have a non-positive complex sectional curvature, $K_{\mathbb{C}} \leqslant 0$, if for every $p \in M$ and every pair $z, w \in T_p^*M \otimes \mathbb{C}$,

$$\langle \mathcal{R}(z \wedge w), \overline{z \wedge w} \rangle \leqslant 0.$$

For example, a manifold with a negative semi-definite curvature operator, $\mathcal{R} \leqslant 0$ (e.g., a hyperbolic manifold, or more generally a symmetric space of non-compact type), has obviously a non-positive complex sectional curvature. A weaker sufficient condition for $K_{\mathbb{C}} \leqslant 0$ is that the (usual) sectional curvature is negative and "pointwise 1/4-pinched", i.e., $-\kappa \leqslant K \leqslant \frac{-\kappa}{4}$ for some positive function κ on M (see [3]).

Proposition 2. *If $K_{\mathbb{C}} \leqslant 0$ then the curvature term in the $\mathrm{Sp}_n\mathrm{Sp}_1$ Bochner formula (see Theorem 1) is $\leqslant 0$.*

Proof. First note that $\mathfrak{g}^\perp \otimes \mathbb{C} = \Lambda_0^2(\mathbf{E}^*) \otimes \boldsymbol{\Sigma}^2$ contains a non-zero decomposable element, e.g., $p_1 \wedge p_2 = z_{12}p^2$. Next, define the following linear functional, T, on the space of curvature type operators:

$$T(\mathcal{R}) = \frac{1}{\mathrm{vol}(G)} \int_G \langle \mathcal{R}(gp_1 \wedge gp_2), \overline{gp_1 \wedge gp_2} \rangle \, d\mu_G.$$

Clearly, $T(\mathcal{R}) \leqslant 0$ if $K_{\mathbb{C}} \leqslant 0$, so it is enough to show that $T(\mathcal{R})$ is a positive constant multiple of $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp)$. Let $\pi : \mathrm{End}(\Lambda^2) \to \mathrm{End}(\mathfrak{g}^\perp)$ be given by $\mathcal{R} \mapsto \mathcal{R}^\perp$, where $\mathcal{R}^\perp$ is the restriction of $\mathcal{R} \in \mathrm{End}(\Lambda^2)$ to $\mathfrak{g}^\perp$ followed by projection onto $\mathfrak{g}^\perp$ (i.e. the "$(\mathfrak{g}^\perp, \mathfrak{g}^\perp)$-block" of $\mathcal{R}$). It is clear, by their definitions, that both $T(\mathcal{R})$ and $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp)$ are G-invariant linear functionals that factor through π. By Schur's lemma, the space of G-invariant linear functionals on $\mathrm{End}(\mathfrak{g}^\perp)$ is 1-dimensional (since $\mathfrak{g}^\perp$ is irreducible). Therefore, $T(\mathcal{R})$ must be a multiple of $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp)$. Evaluating at $\mathcal{R} =$

id_{Λ^2} (the curvature operator of a sphere) we get that $\mathrm{tr}(\mathcal{R}, \mathfrak{g}^\perp) = (\dim \mathfrak{g}^\perp) T(\mathcal{R})$ and the statement follows. □

It follows from this proof that the proposition holds for any orthogonal G such that $\mathfrak{g}^\perp$ is irreducible and $\mathfrak{g}^\perp \otimes \mathbb{C}$ contains a non-zero decomposable 2-form. For example, for $G = \mathrm{U}_n \subset \mathrm{SO}_{2n}$, $n \geqslant 2$ (see also [4, Lemma 4.2]).

Next, we find a natural condition implying $\tau_3 = 0$.

Definition 3. An $\mathrm{Sp}_n\mathrm{Sp}_1$-structure on a manifold is said to be quaternionic-Hermitian if the associated $\mathrm{GL}_n(\mathbb{H})\mathbb{H}^*$-structure is torsionless.

One can identify the intrinsic torsion space for $\mathrm{GL}_n(\mathbb{H})\mathbb{H}^*$ with the subspace $[\mathbf{E}^* \otimes \Lambda^2_0(\mathbf{E}^*)] \otimes \boldsymbol{\Sigma}^3 = W_1 \oplus W_2 \oplus W_3$; thus, an $\mathrm{Sp}_n\mathrm{Sp}_1$-structure is quaternionic-Hermitian if and only if $\tau_1 = \tau_2 = \tau_3 = 0$ ($\tau_2 = \tau_3 = 0$ for $n = 2$).

This condition is attractive also because it turns out to be equivalent to the integrability of the canonical almost complex structure on the twistor space associated with a manifold with an $\mathrm{Sp}_n\mathrm{Sp}_1$-structure (see [7]).

Corollary 2. *A compact quaternionic-Hermitian manifold with non-positive complex sectional curvature is quaternionic-Kähler.*

Proof. This is a consequence of Corollary 1 and Proposition 2. □

A theorem of S.K. Yeung [9] states that *a compact quaternionic-Kähler manifold with negative pointwise* $1/4$*-pinched sectional curvature is a quotient of the quaternionic-hyperbolic space*. Using Corollary 2 we can strengthen this result by relaxing the assumption of “quaternionic-Kähler” to “quaternionic-Hermitian”:

Corollary 3. *A compact quaternionic-Hermitian manifold M with negative* $1/4$*-pinched sectional curvature is a quotient of the quaternionic-hyperbolic space.*

Proof. According to [3], negative $1/4$-pinched sectional curvature implies non-positive complex sectional curvature. Applying Corollary 2 we get that M is quaternionic-Kähler. Now apply the theorem of Yeung. □

Now we apply Corollary 1 to get an analog of Corollary 2 for the closely related manifolds with an Sp_n structure (referred to sometimes as an “almost-hyper-Hermitian” structure).

Definition 4. An Sp_n-structure is said to be hyper-Hermitian if the associated $\mathrm{GL}_n(\mathbb{H})$-structure is torsionless (this is equivalent to the integrability of the three associated almost complex structures I, J, K). A torsionless Sp_n-structure is called hyper-Kähler (this means the 3 complex structures are parallel with respect to the Levi-Civita connection, $\nabla I = \nabla J = \nabla K = 0$).

Corollary 4. *Let M^{4n}, $n \geqslant 2$, be a compact manifold with a hyper-Hermitian structure. If* $\mathrm{tr}(R, \mathfrak{g}^\perp) \leqslant 0$ *then the structure is hyper-Kähler.*

Proof. A hyper-Hermitian Sp_n-structure induces a quaternionic-Hermitian structure, and thus, by Corollary 1, M is quaternionic-Kähler. Now according to Theorem 4.3 of [1], a complex structure compatible with a quaternionic-Kähler structure is necessarily parallel. Apply this to the 3 complex structures I, J, K. □

Corollary 5. *Let M^{4n}, $n \geqslant 2$, be a compact manifold with a hyper-Hermitian structure. If $K_{\mathbb{C}} \leqslant 0$ then the structure is flat (hyper-Kähler with $\mathcal{R} = 0$).*

Proof. By Proposition 2 $\mathrm{tr}(R, \mathfrak{g}^\perp) \leqslant 0$, hence, by the Corollary 4, the structure is hyper-Kähler. This implies that the scalar curvature vanishes [5]. Now non-positive complex sectional curvature, $K_{\mathbb{C}} \leqslant 0$, implies that the (usual) sectional curvature is non-positive, $K \leqslant 0$; but the scalar curvature is an "averaged" sectional curvature, hence $K = 0$, which implies $\mathcal{R} = 0$. □

Remark. The last two corollaries are clearly false in the non-compact case: take the standard (flat) hyper-Kähler structure in $\mathbb{H}^n$, restrict to the open unit ball and change the metric to the hyperbolic metric ($K = -1$). Since this is a conformal change of metric the structure remains hyper-Hermitian, but it is not hyper-Kähler and not flat.

Finally, here is an application with a positive curvature assumption.

Corollary 6. *Let M be a compact 8-dimensional manifold with an $\mathrm{Sp}_2\mathrm{Sp}_1$-structure for which $d\Phi = 0$ and $\mathrm{tr}(R, \mathfrak{g}^\perp) \geqslant 0$ (e.g., if $K_{\mathbb{C}} \geqslant 0$). Then M is quaternionic-Kähler.*

Proof. The condition $d\Phi = 0$ implies $\tau_2 = \tau_5 = \tau_6 = 0$ (see Table 2), i.e., $\tau = \tau_3$, so that the left-hand side of the Bochner formula is non-positive. The condition $\mathrm{tr}(R, \mathfrak{g}^\perp) \geqslant 0$ implies that the right-hand side is non-negative, hence $\tau = 0$. □

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